Section 9.2 Series and Convergence

- **Understand the definition of a convergent infinite series.**
- **Use properties of infinite geometric series.**
- **Use the nth-Term Test for Divergence of an infinite series.**

Infinite Series

One important application of infinite sequences is in representing "infinite summations." Informally, if $\{a_n\}$ is an infinite sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n + \cdot \cdot \cdot \qquad \text{Infinite series}$$

is an **infinite series** (or simply a **series**). The numbers a_1, a_2, a_3, are the **terms** of the series. For some series it is convenient to begin the index at $n = 0$ (or some other integer). As a typesetting convention, it is common to represent an infinite series as simply $\Sigma\, a_n$. In such cases, the starting value for the index must be taken from the context of the statement.

To find the sum of an infinite series, consider the following **sequence of partial sums.**

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n$$

If this sequence of partial sums converges, the series is said to converge and has the sum indicated in the following definition.

INFINITE SERIES

The study of infinite series was considered a novelty in the fourteenth century. Logician Richard Suiseth, whose nickname was Calculator, solved this problem.

If throughout the first half of a given time interval a variation continues at a certain intensity, throughout the next quarter of the interval at double the intensity, throughout the following eighth at triple the intensity and so ad infinitum; then the average intensity for the whole interval will be the intensity of the variation during the second subinterval (or double the intensity).

This is the same as saying that the sum of the infinite series

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdot \cdot \cdot + \frac{n}{2^n} + \cdot \cdot \cdot$$

is 2.

Definitions of Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, the **nth partial sum** is given by

$$S_n = a_1 + a_2 + \cdot \cdot \cdot + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ **converges.** The limit S is called the **sum of the series.**

$$S = a_1 + a_2 + \cdot \cdot \cdot + a_n + \cdot \cdot \cdot$$

If $\{S_n\}$ diverges, then the series **diverges.**

STUDY TIP As you study this chapter, you will see that there are two basic questions involving infinite series. Does a series converge or does it diverge? If a series converges, what is its sum? These questions are not always easy to answer, especially the second one.

Finding the Sum of an Infinite Series **Find the sum of each infinite series. Explain your reasoning.**

a. $0.1 + 0.01 + 0.001 + 0.0001 + \cdot \cdot \cdot$ **b.** $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \cdot \cdot \cdot$

c. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdot \cdot \cdot$ **d.** $\frac{15}{100} + \frac{15}{10{,}000} + \frac{15}{1{,}000{,}000} + \cdot \cdot \cdot$

TECHNOLOGY Figure 9.5 shows the first 15 partial sums of the infinite series in Example 1(a). Notice how the values appear to approach the line $y = 1$.

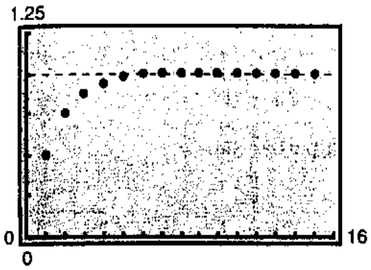


Figure 9.5

NOTE You can geometrically determine the partial sums of the series in Example 1(a) using Figure 9.6.

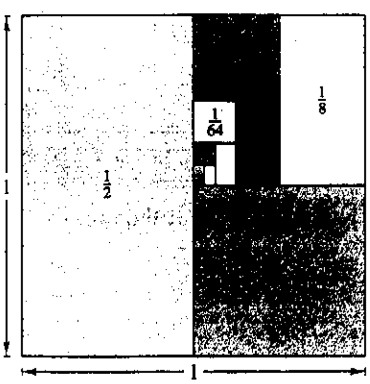


Figure 9.6

EXAMPLE 1 **Convergent and Divergent Series**

a. The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

has the following partial sums.

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

Because

$$\lim_{n\to\infty} \frac{2^n - 1}{2^n} = 1$$

it follows that the series converges and its sum is 1.

b. The nth partial sum of the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

is given by

$$S_n = 1 - \frac{1}{n+1}.$$

Because the limit of S_n is 1, the series converges and its sum is 1.

c. The series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$$

diverges because $S_n = n$ and the sequence of partial sums diverges.

The series in Example 1(b) is a **telescoping series** of the form

$$(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots \qquad \text{Telescoping series}$$

Note that b_2 is canceled by the second term, b_3 is canceled by the third term, and so on. Because the nth partial sum of this series is

$$S_n = b_1 - b_{n+1}$$

it follows that a telescoping series will converge if and only if b_n approaches a finite number as $n \to \infty$. Moreover, if the series converges, its sum is

$$S = b_1 - \lim_{n\to\infty} b_{n+1}.$$

FOR FURTHER INFORMATION To learn more about the partial sums of infinite series, see the article "Six Ways to Sum a Series" by Dan Kalman in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com.*

EXAMPLE 2 Writing a Series in Telescoping Form

Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

Solution

Using partial fractions, you can write

$$a_n = \frac{2}{4n^2 - 1} = \frac{2}{(2n-1)(2n+1)} = \frac{1}{2n-1} - \frac{1}{2n+1}.$$

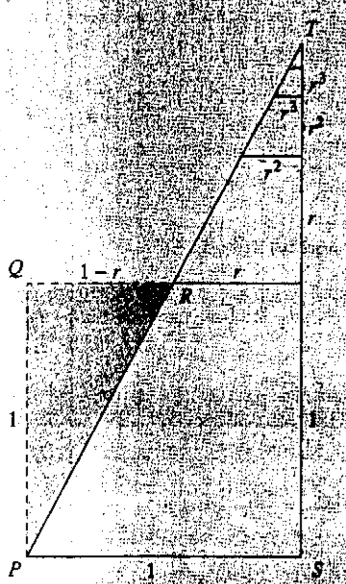
From this telescoping form, you can see that the nth partial sum is

$$S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = 1 - \frac{1}{2n+1}.$$

So, the series converges and its sum is 1. That is,

$$\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{2n+1}\right) = 1.$$

In "Proof Without Words," by Benjamin G. Klein and Irl C. Bivens, the authors present the following diagram. Explain why the final statement below the diagram is valid. How is this result related to Theorem 9.6?



$\Delta PQR \sim \Delta TSP$

$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$

Exercise taken from "Proof Without Words" by Benjamin G. Klein and Irl C. Bivens, *Mathematics Magazine*, October 1988, by permission of the authors.

Geometric Series

The series given in Example 1(a) is a **geometric series.** In general, the series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, \quad a \neq 0 \qquad \text{Geometric series}$$

is a **geometric series** with ratio r.

> **THEOREM 9.6 Convergence of a Geometric Series**
>
> A geometric series with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum
>
> $$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

Proof It is easy to see that the series diverges if $r = \pm 1$. If $r \neq \pm 1$, then $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. Multiplication by r yields

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n.$$

Subtracting the second equation from the first produces $S_n - rS_n = a - ar^n$. Therefore, $S_n(1-r) = a(1-r^n)$, and the nth partial sum is

$$S_n = \frac{a}{1-r}(1 - r^n).$$

If $0 < |r| < 1$, it follows that $r^n \to 0$ as $n \to \infty$, and you obtain

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{a}{1-r}(1-r^n)\right] = \frac{a}{1-r}\left[\lim_{n\to\infty}(1-r^n)\right] = \frac{a}{1-r}$$

which means that the series *converges* and its sum is $a/(1-r)$. It is left to you to show that the series diverges if $|r| > 1$.

TECHNOLOGY Try using a graphing utility or writing a computer program to compute the sum of the first 20 terms of the sequence in Example 3(a). You should obtain a sum of about 5.999994.

EXAMPLE 3 Convergent and Divergent Geometric Series

a. The geometric series

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$$
$$= 3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdot \cdot \cdot$$

has a ratio of $r = \frac{1}{2}$ with $a = 3$. Because $0 < |r| < 1$, the series converges and its sum is

$$S = \frac{a}{1-r} = \frac{3}{1-(1/2)} = 6.$$

b. The geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdot \cdot \cdot$$

has a ratio of $r = \frac{3}{2}$. Because $|r| \geq 1$, the series diverges.

The formula for the sum of a geometric series can be used to write a repeating decimal as the ratio of two integers, as demonstrated in the next example.

EXAMPLE 4 A Geometric Series for a Repeating Decimal

Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

Solution For the repeating decimal $0.\overline{08}$, you can write

$$0.080808\ldots = \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \frac{8}{10^8} + \cdot \cdot \cdot$$
$$= \sum_{n=0}^{\infty} \left(\frac{8}{10^2}\right)\left(\frac{1}{10^2}\right)^n.$$

For this series, you have $a = 8/10^2$ and $r = 1/10^2$. So,

$$0.080808\ldots = \frac{a}{1-r} = \frac{8/10^2}{1-(1/10^2)} = \frac{8}{99}.$$

Try dividing 8 by 99 on a calculator to see that it produces $0.\overline{08}$.

The convergence of a series is not affected by removal of a finite number of terms from the beginning of the series. For instance, the geometric series

$$\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

both converge. Furthermore, because the sum of the second series is $a/(1-r) = 2$, you can conclude that the sum of the first series is

$$S = 2 - \left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3\right]$$
$$= 2 - \frac{15}{8} = \frac{1}{8}.$$

STUDY TIP As you study this chapter, it is important to distinguish between an infinite series and a sequence. A sequence is an ordered collection of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

whereas a series is an infinite sum of terms from a sequence

$$a_1 + a_2 + \cdots + a_n + \cdots.$$

The following properties are direct consequences of the corresponding properties of limits of sequences.

THEOREM 9.7 Properties of Infinite Series

If $\sum a_n = A$, $\sum b_n = B$, and c is a real number, then the following series converge to the indicated sums.

1. $\displaystyle\sum_{n=1}^{\infty} ca_n = cA$
2. $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
3. $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

*n*th-Term Test for Divergence

The following theorem states that if a series converges, the limit of its nth term must be 0.

NOTE Be sure you see that the converse of Theorem 9.8 is generally not true. That is, if the sequence $\{a_n\}$ converges to 0, then the series $\sum a_n$ may either converge or diverge.

THEOREM 9.8 Limit of *n*th Term of a Convergent Series

If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

Proof Assume that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L.$$

Then, because $S_n = S_{n-1} + a_n$ and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = L$$

it follows that

$$\begin{aligned} L = \lim_{n\to\infty} S_n &= \lim_{n\to\infty} (S_{n-1} + a_n) \\ &= \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n \\ &= L + \lim_{n\to\infty} a_n \end{aligned}$$

which implies that $\{a_n\}$ converges to 0.

The contrapositive of Theorem 9.8 provides a useful test for *divergence*. This ***n*th-Term Test for Divergence** states that if the limit of the nth term of a series does *not* converge to 0, the series must diverge.

THEOREM 9.9 *n*th-Term Test for Divergence

If $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

EXAMPLE 5 Using the *n*th-Term Test for Divergence

a. For the series $\sum_{n=0}^{\infty} 2^n$, you have

$$\lim_{n\to\infty} 2^n = \infty.$$

So, the limit of the nth term is not 0, and the series diverges.

b. For the series $\sum_{n=1}^{\infty} \frac{n!}{2n! + 1}$, you have

$$\lim_{n\to\infty} \frac{n!}{2n! + 1} = \frac{1}{2}.$$

So, the limit of the nth term is not 0, and the series diverges.

c. For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, you have

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Because the limit of the nth term is 0, the nth-Term Test for Divergence does *not* apply and you can draw no conclusions about convergence or divergence. (In the next section, you will see that this particular series diverges.)

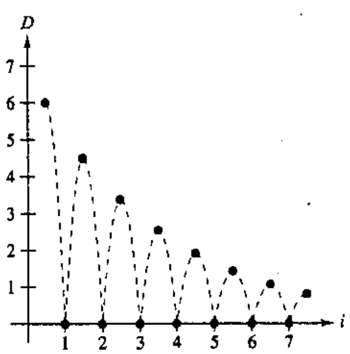
STUDY TIP The series in Example 5(c) will play an important role in this chapter.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

You will see that this series diverges even though the nth term approaches 0 as n approaches ∞.

EXAMPLE 6 Bouncing Ball Problem

A ball is dropped from a height of 6 feet and begins bouncing, as shown in Figure 9.7. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.



The height of each bounce is three-fourths the height of the preceding bounce.
Figure 9.7

Solution When the ball hits the ground for the first time, it has traveled a distance of $D_1 = 6$ feet. For subsequent bounces, let D_i be the distance traveled up and down. For example, D_2 and D_3 are as follows.

$$D_2 = \underbrace{6\left(\tfrac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\tfrac{3}{4}\right)}_{\text{Down}} = 12\left(\tfrac{3}{4}\right)$$

$$D_3 = \underbrace{6\left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)}_{\text{Down}} = 12\left(\tfrac{3}{4}\right)^2$$

By continuing this process, it can be determined that the total vertical distance is

$$\begin{aligned}
D &= 6 + 12\left(\tfrac{3}{4}\right) + 12\left(\tfrac{3}{4}\right)^2 + 12\left(\tfrac{3}{4}\right)^3 + \cdots \\
&= 6 + 12\sum_{n=0}^{\infty}\left(\tfrac{3}{4}\right)^{n+1} \\
&= 6 + 12\left(\tfrac{3}{4}\right)\sum_{n=0}^{\infty}\left(\tfrac{3}{4}\right)^{n} \\
&= 6 + 9\left(\frac{1}{1 - \frac{3}{4}}\right) \\
&= 6 + 9(4) \\
&= 42 \text{ feet.}
\end{aligned}$$

Exercises for Section 9.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the first five terms of the sequence of partial sums.

1. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

2. $\frac{1}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6} + \frac{5}{6 \cdot 7} + \cdots$

3. $3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} - \cdots$

4. $\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots$

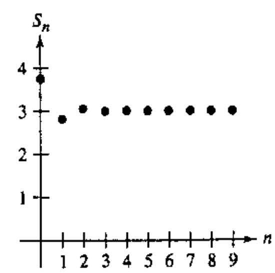
5. $\sum_{n=1}^{\infty} \frac{3}{2^{n-1}}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

In Exercises 7–16, verify that the infinite series diverges.

7. $\sum_{n=0}^{\infty} 3\left(\frac{3}{2}\right)^n$
8. $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n$
9. $\sum_{n=0}^{\infty} 1000(1.055)^n$
10. $\sum_{n=0}^{\infty} 2(-1.03)^n$
11. $\sum_{n=1}^{\infty} \frac{n}{n+1}$
12. $\sum_{n=1}^{\infty} \frac{n}{2n+3}$
13. $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$
14. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$
15. $\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$
16. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

In Exercises 17–22, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).] Use the graph to estimate the sum of the series. Confirm your answer analytically.

(a)


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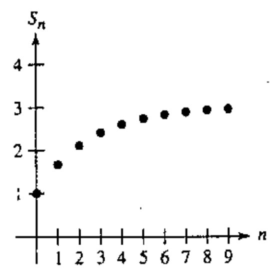

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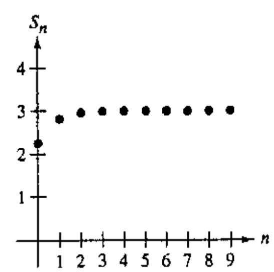

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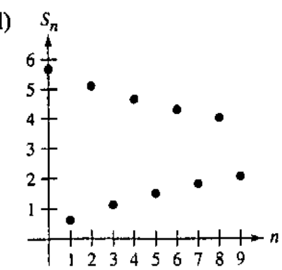

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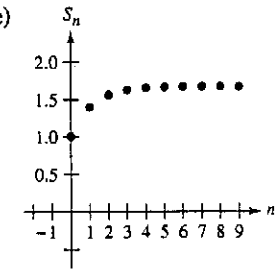
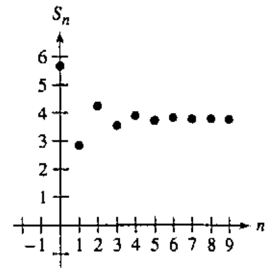


(f)


17. $\sum_{n=0}^{\infty} \frac{9}{4}\left(\frac{1}{4}\right)^n$
18. $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$
19. $\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n$
20. $\sum_{n=0}^{\infty} \frac{17}{3}\left(-\frac{8}{9}\right)^n$
21. $\sum_{n=0}^{\infty} \frac{17}{3}\left(-\frac{1}{2}\right)^n$
22. $\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$

In Exercises 23–28, verify that the infinite series converges.

23. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (Use partial fractions.)

24. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ (Use partial fractions.)

25. $\sum_{n=0}^{\infty} 2\left(\frac{3}{4}\right)^n$
26. $\sum_{n=1}^{\infty} 2\left(-\frac{1}{2}\right)^n$

27. $\sum_{n=0}^{\infty} (0.9)^n = 1 + 0.9 + 0.81 + 0.729 + \cdots$

28. $\sum_{n=0}^{\infty} (-0.6)^n = 1 - 0.6 + 0.36 - 0.216 + \cdots$

Numerical, Graphical, and Analytic Analysis **In Exercises 29–34, (a) find the sum of the series, (b) use a graphing utility to find the indicated partial sum S_n and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum, and (d) explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.**

n	5	10	20	50	100
S_n					

29. $\sum_{n=1}^{\infty} \frac{6}{n(n+3)}$
30. $\sum_{n=1}^{\infty} \frac{4}{n(n+4)}$
31. $\sum_{n=1}^{\infty} 2(0.9)^{n-1}$
32. $\sum_{n=1}^{\infty} 3(0.85)^{n-1}$
33. $\sum_{n=1}^{\infty} 10(0.25)^{n-1}$
34. $\sum_{n=1}^{\infty} 5\left(-\frac{1}{3}\right)^{n-1}$

In Exercises 35–50, find the sum of the convergent series.

35. $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$
36. $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$

37. $\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)}$

38. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)}$

39. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$

40. $\sum_{n=0}^{\infty} 6\left(\frac{4}{5}\right)^n$

41. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$

42. $\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^n$

43. $1 + 0.1 + 0.01 + 0.001 + \cdots$

44. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$

45. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \cdots$

46. $4 - 2 + 1 - \frac{1}{2} + \cdots$

47. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n}\right)$

48. $\sum_{n=1}^{\infty} [(0.7)^n + (0.9)^n]$

49. $\sum_{n=1}^{\infty} (\sin 1)^n$

50. $\sum_{n=1}^{\infty} \frac{1}{9n^2 + 3n - 2}$

In Exercises 51–56, (a) write the repeating decimal as a geometric series and (b) write its sum as the ratio of two integers.

51. $0.\overline{4}$

52. $0.\overline{9}$

53. $0.\overline{81}$

54. $0.\overline{01}$

55. $0.0\overline{75}$

56. $0.2\overline{15}$

In Exercises 57–72, determine the convergence or divergence of the series.

57. $\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$

58. $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$

59. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

60. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$

61. $\sum_{n=1}^{\infty} \frac{3n-1}{2n+1}$

62. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

63. $\sum_{n=0}^{\infty} \frac{4}{2^n}$

64. $\sum_{n=0}^{\infty} \frac{1}{4^n}$

65. $\sum_{n=0}^{\infty} (1.075)^n$

66. $\sum_{n=1}^{\infty} \frac{2^n}{100}$

67. $\sum_{n=2}^{\infty} \frac{n}{\ln n}$

68. $\sum_{n=1}^{\infty} \ln \frac{1}{n}$

69. $\sum_{n=1}^{\infty} \left(1 + \frac{k}{n}\right)^n$

70. $\sum_{n=1}^{\infty} e^{-n}$

71. $\sum_{n=1}^{\infty} \arctan n$

72. $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

Writing About Concepts

73. State the definitions of convergent and divergent series.

74. Describe the difference between $\lim_{n\to\infty} a_n = 5$ and $\sum_{n=1}^{\infty} a_n = 5$.

75. Define a geometric series, state when it converges, and give the formula for the sum of a convergent geometric series.

Writing About Concepts *(continued)*

76. State the nth-Term Test for Divergence.

77. Let $a_n = \frac{n+1}{n}$. Discuss the convergence of $\{a_n\}$ and $\sum_{n=1}^{\infty} a_n$.

78. Explain any differences among the following series.

(a) $\sum_{n=1}^{\infty} a_n$ (b) $\sum_{k=1}^{\infty} a_k$ (c) $\sum_{n=1}^{\infty} a_k$

In Exercises 79–86, find all values of x for which the series converges. For these values of x, write the sum of the series as a function of x.

79. $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$

80. $\sum_{n=1}^{\infty} (3x)^n$

81. $\sum_{n=1}^{\infty} (x-1)^n$

82. $\sum_{n=0}^{\infty} 4\left(\frac{x-3}{4}\right)^n$

83. $\sum_{n=0}^{\infty} (-1)^n x^n$

84. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

85. $\sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n$

86. $\sum_{n=1}^{\infty} \left(\frac{x^2}{x^2+4}\right)^n$

87. (a) You delete a finite number of terms from a divergent series. Will the new series still diverge? Explain your reasoning.

(b) You add a finite number of terms to a convergent series. Will the new series still converge? Explain your reasoning.

88. ***Think About It*** Consider the formula

$$\frac{1}{x-1} = 1 + x + x^2 + x^3 + \cdots.$$

Given $x = -1$ and $x = 2$, can you conclude that either of the following statements is true? Explain your reasoning.

(a) $\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots$

(b) $-1 = 1 + 2 + 4 + 8 + \cdots$

In Exercises 89 and 90, (a) find the common ratio of the geometric series, (b) write the function that gives the sum of the series, and (c) use a graphing utility to graph the function and the partial sums S_3 and S_5. What do you notice?

89. $1 + x + x^2 + x^3 + \cdots$

90. $1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots$

In Exercises 91 and 92, use a graphing utility to graph the function. Identify the horizontal asymptote of the graph and determine its relationship to the sum of the series.

	Function	*Series*
91.	$f(x) = 3\left[\frac{1-(0.5)^x}{1-0.5}\right]$	$\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$
92.	$f(x) = 2\left[\frac{1-(0.8)^x}{1-0.8}\right]$	$\sum_{n=0}^{\infty} 2\left(\frac{4}{5}\right)^n$